A CERTAIN MODIFICATION OF THE FOURIER HYPOTHESIS

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We construct impulse solutions for the heat transfer problems.

The majority of works concerned with the study of the processes of heat transfer are based on the classical heat conduction equation, which in turn results from the Fourier hypothesis:

$$Q = -k\frac{\partial T}{\partial x}. \tag{1}$$

In such a case it is often emphasized that in solving Eq. (1) the velocity of propagation of disturbances is assumed to be infinite. This permits one to obtain correct results in many cases, but not always.

Recently, time publications appeared suggesting that relation (1) be modified on the basis of the Maxwell hypothesis [1, 2]:

$$Q_{|t+\tau} = -k\frac{\partial T}{\partial x}\Big|_t. \tag{2}$$

We introduce relation (2) to take into account the finite velocity of propagation of thermal disturbances.

In fact, let us consider the equation

$$\rho c_\vartheta\frac{\partial T}{\partial t} = -\frac{\partial Q}{\partial x} \tag{3}$$

and take the zero approximation of Eq. (2) at rather small values of τ, $\tau > 0$, i.e.,

$$Q + \tau\frac{\partial Q}{\partial t} = -k\frac{\partial T}{\partial x}. \tag{4}$$

Then, relations (3) and (4) yield [1, 2]

$$\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2} + \frac{k}{\rho c_\vartheta}\frac{\partial^2 T}{\partial x^2} = 0 \tag{5}$$

and thus we determine the velocity of propagation of the temperature front $V = dx/dt = k/\tau\rho c_V$ [3].

We note that on the same grounds of accounting for the finite velocity of propagation of heat, it is necessary to consider the relation (equation)

$$\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2} - \tau\frac{\partial}{\partial x}\left(\lambda k\frac{\partial T}{\partial x}\right) = 0, \tag{6}$$

where λ is a certain characteristic of the medium. When $\lambda = k/(\tau\rho c_V)$, Eqs. (5) and (6) coincide.

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Within the framework of relation (2) this means that besides the delay τ a certain parameter $\delta \in R$ should exist such that $V_p = \delta/\tau$, where V_p is the velocity of propagation of disturbances of the heat flux.

Let us consider the following modification of hypothesis (2):

$$Q(x+\delta,\ t+\tau) = -k\frac{\partial T}{\partial x}(x,\ t), \tag{7}$$

where $\delta, \tau > 0$ are sufficiently small numbers. The physical meaning of this equality can be represented as follows: we perform the shift $x \to x - \delta$ and write down Eq. (7) in a zero approximation:

$$Q(x,\ t+\tau) = -k\frac{\partial T}{\partial x}(x,\ t) + k\delta\frac{\partial^2 T}{\partial x^2} + O(\delta^2). \tag{8}$$

Consequently, in the time τ the heat flux responds not only to the change in the temperature gradient, but also to the "diffusion of the temperature front." Thus, at $\delta = 0$ it follows from Eq. (8) that the existence of the temperature T at a certain time t at the point x does not as yet mean the existence of the flux Q, which appears only at the instant of time $t + \tau$. However, since implicitly we assume the presence of the heat flux, then in the time τ it will be displaced from the point (x, t) to the point $(x + \delta, t')$, i.e., it is possible to consider the relations

$$\rho c_\vartheta \frac{\partial T}{\partial t}(x,\ t) = -\frac{\partial Q}{\partial x}(x,\ t), \tag{9}$$

$$Q(x+\delta,\ t+\tau) = -k\frac{\partial T}{\partial x}(x,\ t), \tag{10}$$

which later will be replaced by other equations according to the following scheme. We expand the function Q into a Taylor series accurate to values of the first order of smallness:

$$Q + \tau\frac{\partial Q}{\partial t} + \delta\frac{\partial Q}{\partial x} = -k\frac{\partial T}{\partial t}. \tag{11}$$

Since $\delta = v\tau$, we seek the heat flux component in the form: $Q(\cdot) = \hat{Q}(t - x/v)$, i.e., the quantity Q has the form of a stationary wave travelling with the velocity $dx/dt = V$, and relation (11) assumes the form

$$\hat{Q}(x,\ t) = -k\frac{\partial T}{\partial x}(x,\ t). \tag{12}$$

We note that equality (12) does not contradict hypothesis (7), since the family of the self-similar functions Q, determined above, is invariant with respect to the shift $(x, t) \to (x, \delta, t + \tau)$. In fact,

$$Q(x+\delta,\ t+\tau) \overset{\text{def}}{=} \hat{Q}\left(t+\tau-\frac{x+\delta}{V}\right) = \hat{Q}\left(t-\frac{x}{V}\right) \overset{\text{def}}{=} Q(x,\ t).$$

Then, Eq. (12) yields the relation

$$Q(x+\delta,\ t+\tau) = -k\frac{\partial T}{\partial x}(x,\ t).$$

which coincides with equality (7)

Thus, Eq. (10) is invariant with respect to the indicated shift if and only if the heat flux satisfies the equation

$$\frac{\partial Q}{\partial t} + V_p\frac{\partial Q}{\partial x} = 0, \tag{13}$$

but this is equivalent to claiming that we can consider the system of equations

$$Q = -k\frac{\partial T}{\partial x},$$
$$\rho c_\vartheta \frac{\partial T}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial t} + V_p\frac{\partial Q}{\partial x} = 0. \tag{14}$$

We note that Eqs. (9) and (10) yield the system (14), but, generally speaking, the inverse is incorrect, i.e., we have "lost" a portion of the solutions. However, the solutions that remain (simple waves) have rather interesting properties.

It can also be trivially verified that the function $\overline{Q}(t + x/v)$ leaves the following equation invariant with respect to the shift:

$$Q(x-\delta, t+\tau) = -k\frac{\partial T}{\partial x}(x, t). \tag{15}$$

Then, system (9), (10), (15) is equivalent to Eqs. (14), where the parameter $V_p = -V$, i.e., for fluxes moving with the velocity V in both the forward and reverse directions we have

$$\frac{\partial \hat{Q}}{\partial t} + V_p\frac{\partial \hat{Q}}{\partial x} = 0,$$
$$\frac{\partial \overline{Q}}{\partial t} - V_p\frac{\partial \overline{Q}}{\partial x} = 0, \tag{16}$$

where $\hat{Q}$ and $\overline{Q}$ are the direct and reverse waves. To obtain such solutions, it is sufficient to note that the last equation of system (14) yields the wave equation

$$\frac{\partial^2 Q}{\partial t^2} - V_p^2\frac{\partial^2 Q}{\partial x^2} = 0, \tag{17}$$

which is verified by direct differentiation. As is known, solutions of Eq. (17) have the form $Q = \hat{Q} + \overline{Q}$ and thus include the solutions of Eqs. (16).

Consequently, we may assume that the system of equations (9), (10), and (14) is equivalent to the system

$$\rho c_\vartheta\frac{\partial T}{\partial t} - \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial^2 Q}{\partial t^2} - V_p^2\frac{\partial^2 Q}{\partial x^2} = 0, \quad Q = -k\frac{\partial T}{\partial x}, \tag{18}$$

involving the wave equation. This is basically the main result of the present work.

The solution of the wave equation can be reduced to the solution of a well-known problem [3] in the following way. Denoting $\partial Q/\partial t = J$ and $\partial Q/\partial x = -U$, we obtain the system

$$\frac{\partial J}{\partial t} + V_p^2\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial t} + \frac{\partial J}{\partial x} = 0, \quad (x, t) \in [0, l] \times \mathbf{R}^+, \quad l > 0, \tag{19}$$

for which (by analogy with the problem of [3]) we consider the boundary conditions

$$J_{|x=0} = 0; \quad J = \Phi(U)_{|x=l}. \tag{20}$$

where $\Phi : R^1 \to R^1$ is a certain function, and the initial conditions

$$J(x, 0) = J_0(x); \quad U(x, 0) = U_0(x). \tag{21}$$

In system (19) we perform the substitution of variables: $J = 0.5(u + v)$ and $U = z/2(u - v)$, $x = ls$, $t = (l/V_p)z$, where $z^2 = 1/V_p^2$. As a result we obtain [3]

$$\frac{\partial u}{\partial \bar{t}} + \frac{\partial u}{\partial s} = 0\,, \quad \frac{\partial v}{\partial \bar{t}} - \frac{\partial v}{\partial s} = 0\,, \quad (s\,,\,\bar{t}) \in [0\,,\,1] \times R^+\,, \tag{22}$$

with the boundary conditions $u = -v|_{s=0}$, $(-v) = f(u)|_{s=1}$, where f is a function assigned implicitly by the relation

$$\frac{z}{2}(u + f) = \Phi\left(\frac{u - f}{2}\right),$$

and the initial conditions

$$u(s\,,\,0) = \varphi_1(s)\,, \quad v(s\,,\,0) = \varphi_2(s)\,,$$

where

$$\varphi_i(s) = J_0(ls) + \frac{(-1)^{i-1}}{z} U_0(ls)\,, \quad i = 1\,,\,2\,.$$

Since the functions u and v do not change their values along the characteristics $ds/dt = \pm 1$, respectively, the solution can be represented in the form

$$u(s\,,\,\bar{t}) = y(\bar{t} - s)\,, \quad v(s\,,\,t) = y(\bar{t} + s)\,,$$

where, by virtue of the boundary conditions, $y(t)$ is the solution of the difference equation

$$y(\bar{t} + 2) = f(y(\bar{t}))\,, \quad \bar{t} \in [-1\,,\,+\infty)\,,$$

with the initial function [3]

$$y(\bar{t})|_{t \in [-1,1)} = \begin{cases} u_0(\bar{t})\,, & \bar{t} \in [-1\,,\,0)\,, \\ v_0(\bar{t})\,, & \bar{t} \in [0\,,\,1)\,. \end{cases}$$

Thus, the initial problem has been reduced to a difference equation, which can be solved by the method of iterations. It is known [3] that the solutions of such a problem for $t \to \infty$ tend to piecewise-constant periodic functions that assume values from $\mathbf{P}^+$, where $\mathbf{P}^+$ is the set of attracting motionless points of the mapping f, i.e.,

$$\mathbf{P}^+ \overset{\text{def}}{=} \left\{ \xi = f(\xi) \setminus |f(\xi)| < 1 \right\}.$$

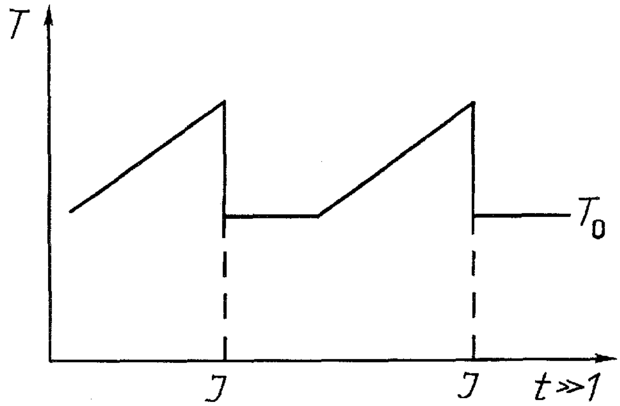
Next, using the derivative

$$\lim_{t \to \infty} \frac{\partial T}{\partial t}(x\,,\,t) = \frac{\mathbf{P}^+}{\rho c_\vartheta V_p} = \tan\theta\,, \quad x \in (0\,,\,l)\,,$$

where θ is the angle of inclination of the graph T at rather large times and a fixed value of $x \in (0, l)$, it is not difficult to find that

$$T - T_0 \simeq \Omega_{(\partial T/\partial t)} \tan\theta = \frac{\mathbf{P}^+}{V_p} \cdot 2^m \frac{l}{V_p} = 2^m \mathbf{P}^+ \frac{l}{V_p^2}\,. \tag{23}$$

Here T is a certain fixed level, and the sign $\simeq$ means an approximate order of the magnitude, since the amplitude is determined accurately by the set τ of discontinuity points of the limiting solution (Fig. 1). Figure 1 shows the



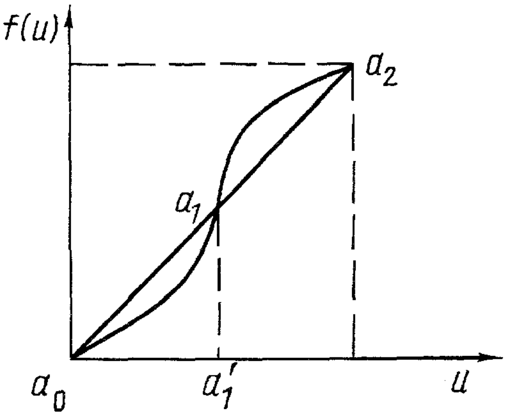


Fig. 1. Limiting distribution for the wave equation.

Fig. 2. Form of the function "generating" relaxational vibrations.

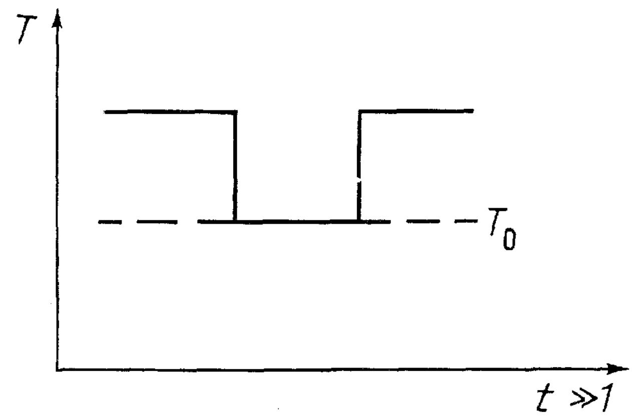


Fig. 3. Limiting distribution for the first-order system.

case where the mapping has the form given in Fig. 2, where $f(\hat{J}) < \hat{J}$, $\hat{J}$ is a certain open bounded interval; a_0, $a_2 \in \mathbf{P}^+$, a_1 is the repelling motionless point of f, whose inverse images, under the action of iterations, determine the set of discontinuity points τ [3].

Thus, $\rho c_v \partial T/\partial t \to P^+/V_p$ for $t \to \infty$ uniformly for almost all $x \in (0, l)$, where $\mathbf{P}^+ = \{a_0, a_2\}$; the period $\Omega(\partial T/\partial t) = 2\,\mathbf{P}^+ l/(\rho c_v V_p^2)$. For example, if the inverse image of a_0, i.e., the point a_1 in Fig. 2, divides the interval $\hat{J}$ in half, then to determine the amplitude it is necessary in equality (23) to take $\Omega_{(\cdot)}/2$ instead of $\Omega_{(\partial T/\partial t)} = P^+/(\rho c_v V_p{}^2)$, i.e., $T - T_0 = (a_2/V_p)2(1/2)(l/V_p)$. The general case is similar.

In conclusion we note that in [4] heat conduction equations are considered for $\delta = 0$ and $\tau > 0$ with the boundary conditions

$$Q = 0_{|x=0}; \quad Q = \hat{\Phi}(T)_{|x=l},$$

where, for example [5],

$$\hat{\Phi} \overset{\text{def}}{=} q(T) - \varepsilon E(T)\, T^k, \quad k \geq 4.$$

Here $E(T)$ is a function that characterizes the absorbing properties of the medium, $q(\cdot)$ is a certain incident flux; in the simplest case $Q = q(T)$.

We may assume that in the case of intense external effects we have the boundary condition (given only as an illustration)

$$\frac{\partial Q}{\partial t} = q(T)\left.\frac{\partial T}{\partial t}\right|_{x=l}.$$

which follows from condition (20) and the first relation of system (17). In [4] the temperature distributions shown in Fig. 3 are obtained. A comparison of Figs. 1 and 3 shows that in the case of rather strong external effects the periodic piecewise-constant heat wave tends toward "inverting."

NOTATION

Q, heat flux; T, temperature; k, thermal conductivity; x, spatial coordinate; t, time; τ, time of heat flux relaxation; ε, Stefan-Boltzmann constant.

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